Announcements

1) Can turn in HW 1 until 8 today

2) HW 2 up on Canvas, due next Thursday

Generating Sets

Let V be a vector space over $\mathbb{F}$. A subset S of V is called generating (or spanning) for V if $\mathrm{span}(S) = V$ (span is over $\mathbb{F}$).

Example 1:

(homogeneous systems of linear equations)

Consider a system of homogeneous linear equations

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = 0$$
$$a_{2,1}X_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = 0$$
$$\vdots$$
$$a_{m,1}X_1 + a_{m,2}X_2 + \cdots + a_{m,n}X_n = 0$$

where $a_{i,j}$ is either real or complex for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

The solution space to the system is all vectors $(y_1, y_2, \ldots, y_n)$ in either $\mathbb{R}^n$ or $\mathbb{C}^n$ such that

$$\boxed{\sum_{j=1}^{n} a_{i,j} y_j = 0} \text{ for all } 1 \leq i \leq m$$

The solution set is a subspace of $\mathbb{R}^n$ or $\mathbb{C}^n$ (use subspace test).

A generating set can be obtained for the solution set - but I need words I haven't defined yet!

Example 2: (frames in $\mathbb{R}^n$)

A finite subset

$S = \{w_1, w_2, \dots, w_k\}$

of $\mathbb{R}^n$ that is spanning

and satisfies the

frame condition:

$\exists\ A, B \in \mathbb{R}^+$ such that

$\forall\ v \in \mathbb{R}^n$,

$$\boxed{A\, v \cdot v \leq \sum_{i=1}^{k} |v \cdot w_i|^2 \leq B\, v \cdot v}$$

Here "$v \cdot u$" is the usual dot product on $\mathbb{R}^n$.

If $v = (v_1, \dots, v_n)$

$u = (u_1, \dots, u_n)$,

$$v \cdot u = \sum_{i=1}^{n} (v_i u_i) .$$

BMW frame

$$S \subseteq \mathbb{R}^2$$

$$S = (w_1, w_2, w_3)$$

where

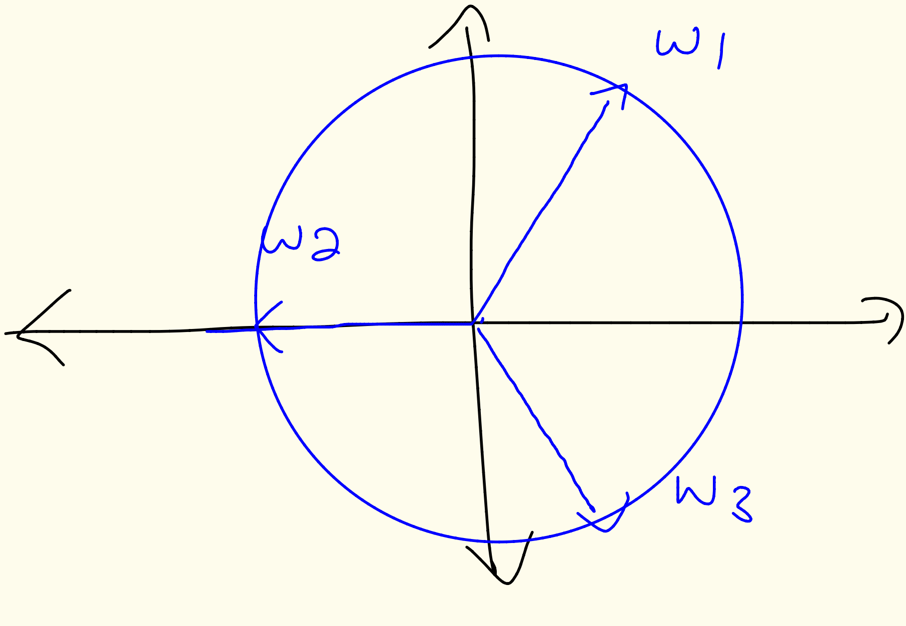
$$w_1 = (\cos(60^\circ), \sin(60^\circ))$$
$$w_2 = (\cos(180^\circ), \sin(180^\circ))$$
$$w_3 = (\cos(300^\circ), \sin(300^\circ))$$

$$w_1 = \left(1/2, \sqrt{3}/2\right)$$

$$w_2 = (-1, 0)$$

$$w_3 = \left(1/2, -\sqrt{3}/2\right)$$

Picture



$$w_1 - w_3 = (0, \sqrt{3})$$

$$w_2 = (-1, 0)$$

so if $(\alpha, \beta) \in \mathbb{R}^2$,

$$(\alpha, \beta) = -\alpha w_2 + \frac{\beta}{\sqrt{3}}(w_1 - w_3).$$

This shows S is spanning for $\mathbb{R}^2$.

Frame Condition

$$v = (v_1, v_2, v_3)$$

$$|v \cdot w_1| = \left|\frac{v_1 + \sqrt{3}\,v_2}{2}\right|$$

$$|v \cdot w_3| = \left|\frac{v_1 - \sqrt{3}\,v_2}{2}\right|$$

$$|v \cdot w_2| = |v_1|$$

Squaring each term and adding yields

$$\frac{3v_1^2}{2} + \frac{3v_2^2}{2}, \text{ so}$$

we could choose $A = B = 3/2$.

Example 3: (non-spanning set)

$S=\{(1,2,4),(-1,0,1)\}$

is not a spanning set for $\mathbb{R}^3$.

Matrix techniques show this.

Linear Independence

(Section 1.5)

Notation: $(e_i^n)_{i=1}^n$

e_i^n is the vector in $\mathbb{R}^n$ with a one in the i^{th} position and zeros in all other positions.

If we fix any i, $1 \leq i \leq n$, then e_i^n is not in the span of $\{e_j^n\}_{\substack{1 \leq j \leq n \\ i \neq j}}$ because all of these vectors have zero in the i^{th} position

Definition: (linear independence)

Let V be a vector space over $\mathbb{F}$. A nonempty subset S of V is called **linearly independent** (over $\mathbb{F}$) if

$\forall\ x_1, x_2, \ldots, x_n \in S$, $x_i \neq 0_V\ \forall\ 1 \leq i \leq n$,

then if $\boxed{\sum_{i=1}^{n} \alpha_i x_i = 0_V}$ for some scalars $\alpha_1, \ldots, \alpha_n$ then

$\boxed{\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0_{\mathbb{F}}}$.

Note: this is equivalent to saying that whenever you take $x \in S$, $x \neq 0_V$, then $x \notin \mathrm{Span}(S \setminus \{x\})$.

We have already shown that $(e_i^n)_{i=1}^n$ is linearly independent over $\mathbb{R}$. When no confusion arises, we'll drop the superscript "n" and just write $(e_i)_{i=1}^n$.

Example 4: (matrix units)

In $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$, if we let $e_{i,j}$ be the matrix with a one in the (i,j) position and zeros elsewhere, then $\{e_{i,j}\}_{i,j=1}^{n}$ is linearly independent over $\mathbb{R}$ or $\mathbb{C}$.

Example 5: (monomials)

Let V be the vector space of polynomials with real coefficients (over $\mathbb{R}$). The set $S = \{x^n \mid n \geq 0\}$ $(n \in \mathbb{N})$

is linearly independent over $\mathbb{R}$.

If $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{R}$

and $\sum_{i=0}^{n} \alpha_i x^i = 0$,

then as a function on $\mathbb{R}$,

$f(x) = \sum_{i=0}^{n} \alpha_i x^i$ is the zero

function. This

implies $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$

since x can be any real

number.

By the fundamental theorem of algebra, f has at most n zeros, contradiction.

Therefore the monomials $\{x^n\}_{n=0}^{\infty}$ are linearly independent over $\mathbb{R}$.

Example 6 (linearly dependent set)

Let x be any vector in a vector space V over a field $\mathbb{F}$.

Then if α is any scalar, $x \neq 0_V$, $\alpha \neq 1_{\mathbb{F}}, 0_{\mathbb{F}}$

then $\{x, \alpha x\}$ is linearly dependent.

Bases and Dimension

(section 1.6)

A **basis** for a vector space V over a field $\mathbb{F}$ is a **maximal** linearly independent subset of V.

Maximal = not contained in any larger linearly independent set.

Consequence: any basis for V over $\mathbb{F}$ must be spanning for V.